Singapore Mathematical Society

Singapore Secondary School Mathematical Olympiad 1992

Part A

Saturday, 30 May 1992 1000–1200

Attempt as many questions as you can.
No calculators are allowed.
Enter your answers on the answer sheet provided.
No steps are needed to justify your answers.
Each question carries 3 marks.

1. Let a, b, c, and d be real numbers satisfying

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{a}.$$

Find the positive value of $\dfrac{a+b+c+d}{a+b+c-d}$.

(A) 0 (B) 2 (C) 3 (D) 4 (E) None of the preceding.

2. Let f be a function such that

$$f(x+y) = f(x)f(y)$$

for any real numbers x and y. If $f(1) = \frac{1}{16}$, then the value of $f(-1)$ is

(A) $-\frac{1}{16}$ (B) 16 (C) $\frac{1}{16}$ (D) -16 (E) 0.

3. The probabilities that three hunters independently hit a rabbit are respectively $\frac{3}{5}$, $\frac{3}{10}$, and $\frac{1}{10}$. What is the probability that the rabbit will be shot if they shoot at it simultaneously?

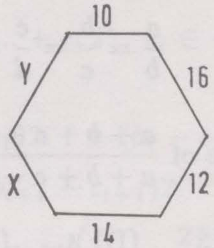
(A) $\dfrac{9}{500}$ (B) $\dfrac{126}{500}$ (C) $\dfrac{374}{500}$ (D) $\dfrac{27}{500}$ (E) None of the preceding.

4. If $x = \dfrac{-1+\sqrt{17}}{2}$, then the value of $x^3 + 2x^2 - 3x - 4$ is

(A) $\dfrac{-1+17\sqrt{17}}{8}$ (B) $\dfrac{17\sqrt{17}}{8}$ (C) $\dfrac{17\sqrt{17}}{8} - 4$ (D) 0 (E) 4.

5. Given that the sum of squares of two numbers is 468, and the sum of their least common multiple and their highest common factor is 42, find the larger number of these two numbers.

6. The sides of an equiangular hexagon measure 10, 16, 12, 14, x and y units, in the order given as shown in the following figure. Find x and y.



7. Suppose the equation

$$x(x+1) + (x+1)(x+2) + \ldots + (x+n-1)(x+n) = 10n$$

has roots r and $r+1$. Find n and r.

8. Ten distinct points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some or all of these ten points as vertices?

9. Let x be the decimal part of $2+\sqrt{3}$. Find the value of $\dfrac{x^2}{1-x}$.

10. What is the remainder when x^{24} is divided by x^2+x+1?

11. The three-digit number $2x3$ is added to the number 326 to give a three-digit number $5y9$. Suppose $5y9$ is divisible by 9. Find the value of $x+y$.

12. The number $2^{48}-1$ is exactly divisible by two numbers between 60 and 70. Find these two numbers.

13. Find the integral value of $(52+6\sqrt{43})^{\frac{3}{2}}-(52-6\sqrt{43})^{\frac{3}{2}}$.

14. Find the integral value of $\sqrt{998\times 999\times 1000\times 1001+1}$.

15. How many pairs of positive integers (x,y) are there that satisfy the equation $xy-2y+1=0$?

16. Determine the value of q so that the equation $x^6+px^4+qx^2-225=0$ has six real roots in arithmetic progression.

17. Let $n! = n(n-1)(n-2)...3 \cdot 2 \cdot 1$ for any positive integer n. Suppose x, y, z and w are positive integers such that

$$w! = x! + y! + z!.$$

Find the value of w.

18. Let $ABCD$ be a unit square. Suppose E is a point on CD with $CE : CD = 3 : 4$. Let F be the intersection of BE and AC. Find the length of EF.

19. A parallelogram has consecutive sides with lengths 9 and 7 and diagonals of integral lengths. How long are these diagonals?

20. In a triangle ABC, $AB = 2BC$ and $\angle B = 2\angle A$. Find $\angle C$.

—E N D—

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Part B

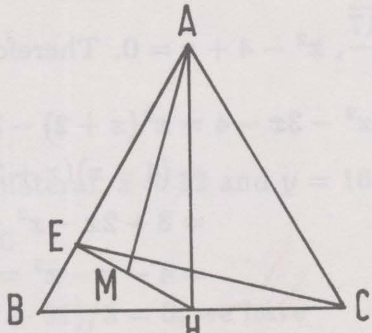
Saturday, 30 May 1992 1200–1300

Attempt as many questions as you can.
No calculators are allowed.
Each question carries 20 marks.

1. Let $a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}$ be distinct real numbers. Suppose a and b are real numbers satisfying the following property:
For each a_i, $1 \leq i \leq n+m$, then either $a_i + a = a_j$ for some j such that $1 \leq j \leq n$, or $a_i - b = a_j$ for some j such that $n+1 \leq j \leq n+m$.

Prove that $na = mb$.

2. In the isosceles triangle ABC, with $AB = AC$, let H be the foot of the altitude from A, let E be the foot of the perpendicular from H to AB, and let M be the mid-point of EH. Show that $AM \perp EC$.



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Part A
Solutions

1. Let $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d} = \dfrac{d}{a} = k$. Then $a = bk = ck^2 = dk^3 = ak^4$. Since $a \neq 0$, $k = \pm 1$. When $k = 1$, $\dfrac{a+b+c+d}{a+b+c-d} = 2$; when $k = -1$, $\dfrac{a+b+c+d}{a+b+c-d} = 0$.

Ans: 2

2. $f(1+0) = f(1)f(0)$. Hence $f(0) = 1$. Since $f(1-1) = f(1)f(-1)$, $f(-1) = 16$.

Ans: 16

3. The required probability $= 1 - \left(\frac{2}{5}\right)\left(\frac{7}{10}\right)\left(\frac{9}{10}\right)$
$= \dfrac{374}{500}$.

Ans: $\dfrac{374}{500}$

4. Since $x = \dfrac{-1+\sqrt{17}}{2}$, $x^2 - 4 + x = 0$. Therefore

$$\begin{aligned} x^3 + 2x^2 - 3x - 4 &= x^2(x+2) - 3x - 4 \\ &= (4-x)(x+2) - 3x - 4 \\ &= 8 + 2x - x^2 - 3x - 4 \\ &= 4 - x - x^2 = 0 \end{aligned}$$

Ans: 0

5. Let the two numbers be x and y, and their LCM $(x, y) = d$. Then

$$x^2 + y^2 = 468 \qquad (1)$$

$$d + \frac{xy}{d} = 42 \qquad (2)$$

Let $x = dx_1$, $y = dy_1$. Then $(x_1, y_1) = 1$. Hence

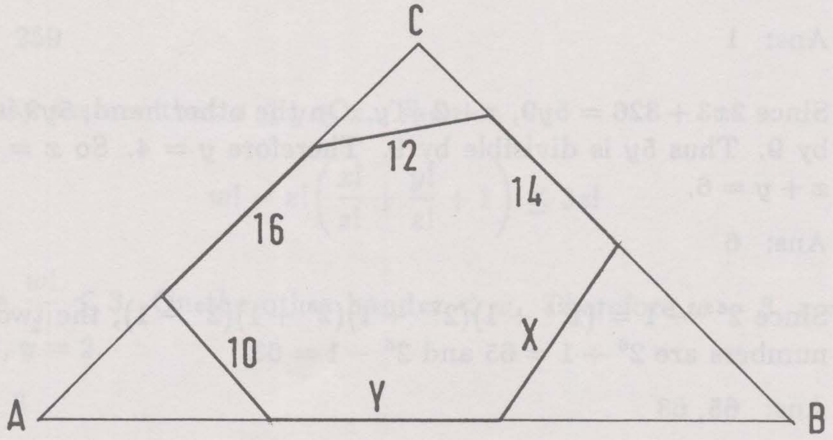
$$x_1^2 + y_1^2 = \frac{468}{d^2} \tag{3}$$

$$1 + x_1 y_1 = \frac{42}{d} \tag{4}$$

Thus $d = 1, 2, 3$ or 6. We can check easily that only when $d = 6$, $(x_1, y_1) = 1$. In this case $x_1 = 2$, $y_1 = 3$. Hence $x = 12$, $y = 18$.

Ans: 18

6.



Since ΔABC is equilateral, $x = 12$ and $y = 16$.

Ans: $x = 12$, $y = 16$

7. Since $x^2 + nx + (n^2 - 31)/3 = 0$, we have

$$2r + 1 = -n \quad \text{and} \quad r(r+1) = (n^2 - 31)/3.$$

Hence $n = 11$, $r = -6$.

Ans: $n = 11$, $r = -6$

8. The number of convex polygons $= \binom{10}{3} + \binom{10}{4} + \ldots + \binom{10}{10}$

$= 2^{10} - \binom{10}{0} - \binom{10}{1} - \binom{10}{2} = 968.$

Ans: 968

9. Since $2+\sqrt{3} = 3+(\sqrt{3}-1)$ and $0 < \sqrt{3}-1 < 1$, $x = \sqrt{3}-1$. Hence $\dfrac{x^2}{1-x} = 2.$

Ans: 2

10. Since $x^{24} - 1 = (x^2 + x + 1)(x - 1)(x^3 + 1)(x^6 + 1)(x^{12} + 1)$, the remainder is 1.

Ans: 1

11. Since $2x3 + 326 = 5y9$, $x + 2 = y$. On the other hand, $5y9$ is divisible by 9. Thus $5y$ is divisible by 9. Therefore $y = 4$. So $x = 2$. Hence $x + y = 6$.

Ans: 6

12. Since $2^{48} - 1 = (2^{24} + 1)(2^{12} + 1)(2^6 + 1)(2^6 - 1)$, the two required numbers are $2^6 + 1 = 65$ and $2^6 - 1 = 63$.

Ans: 65, 63

13. Since $52 + 6\sqrt{43} = 43 + 6\sqrt{43} + 9 = (\sqrt{43} + 3)^2$ and $52 - 6\sqrt{43} = (\sqrt{43} - 3)^2$, we have

$$(52 + 6\sqrt{43})^{3/2} - (52 - 6\sqrt{43})^{3/2} = 828.$$

Ans: 828

14. Since

$$\begin{aligned}\sqrt{(k-2)(k-1)k(k+1)+1} &= \sqrt{(k^2-k)^2 - 2(k^2-k) + 1} \\ &= k^2 - k - 1,\end{aligned}$$

the required value $= 1000^2 - 1000 - 1 = 998999$

Ans: 998999

15. Since $xy - 2y + 1 = 0$ and $y > 0$, $\frac{1}{y} = 2 - x$. Hence $2 > x$. Thus $x = 1$ and $y = 1$.

Ans: 1

16. Observe that 0 is not a root and if β is a root, then $-\beta$ is also a root. Let the smallest positive root be α. Then the six roots are -5α, -3α, $-\alpha$, α, 3α and 5α. Hence $(-5\alpha)(-3\alpha)(-\alpha)(\alpha)(3\alpha)(5\alpha) = -225$. Thus $\alpha = 1$, and the six roots are $-5, -3, -1, 1, 3, 5$. The equation can be written as

$$(x^2 - 1)(x^2 - 9)(x^2 - 25) = 0.$$

Hence $q = 259$.

Ans: 259

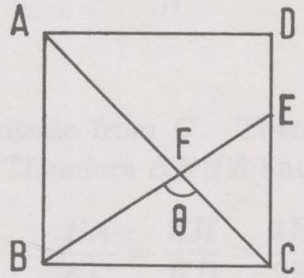
17. We may assume that $x \le y \le z$. Then

$$w! = z!\left(\frac{x!}{z!} + \frac{y!}{z!} + 1\right) \le 3z!$$

Hence $\frac{w!}{z!} \le 3$. On the other hand $z < w$. Therefore $w = 3$, $z = 2$, $x = 2, y = 2$.

Ans: 3

18.



By sine rule, $\frac{BF}{BC} = \frac{\sin 45°}{\sin\theta} = \frac{\sin 45°}{\sin(180° - \theta)} = \frac{FE}{CE}$. So

$$\frac{BF}{FE} = \frac{BC}{CE} = \frac{CD}{CE} = \frac{4}{3}.$$

Hence $FE = \frac{3}{4}BF = \frac{3}{7}BE$.

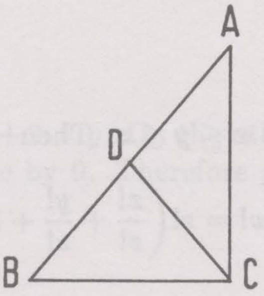
Since $BE = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$, $FE = \frac{15}{28}$.

Ans: $\frac{15}{28}$

19. Let the length of diagonals be c and d. Then $2(9^2 + 7^2) = c^2 + d^2$. The integral pairs whose squares total 260 are $(16, 2)$ and $(14, 8)$. The triangular inequality rules out $(16, 2)$.

Ans: 14, 8

20.



Let D be the point on AB such that $\angle ACD = \angle A$. Then $\angle CDB = 2\angle A$. Thus $\angle CDB = \angle B$. Hence $BC = CD = DA$. Note that $AB = 2BC$, so we have $BD = BC = CD$. Hence ΔBCD is equilateral. Therefore $\angle C = 90°$.

Ans: $90°$

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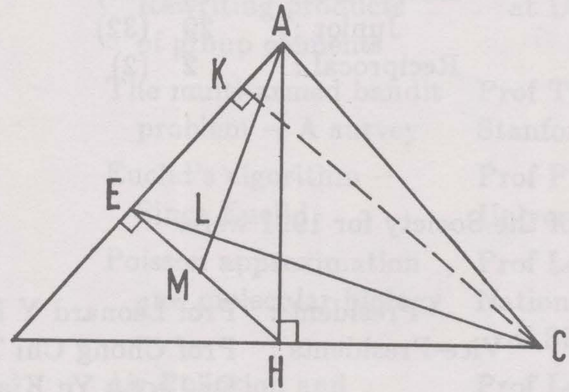
Part B

Solutions

1. Let $b_1 = a_1 - a$, $b_2 = a_2 - a, \ldots, b_n = a_n - a$ and $b_{n+1} = a_{n+1} + b$, $b_{n+2} = a_{n+2} + b, \ldots, b_{n+m} = a_{n+m} + b$. Then each a_i, $1 \le i \le n+m$, is one of the b_j, $1 \le j \le n+m$. Note that a_i's are all distinct and there are $n+m$ b_j's. Therefore b_j's are all distinct and $b_1 + b_2 + \ldots + b_{n+m} = a_1 + a_2 + \ldots + a_{n+m}$. Hence $na = mb$.

2.



Let CK be the altitude from C. Then $CAKH$ is cyclic. Hence $\angle KAH = \angle KCH$. Therefore ΔEHA and ΔKBC are similar. Thus

$$\frac{EA}{KC} = \frac{EH}{KB} = \frac{2EM}{2KE}.$$

Consequently, $\angle EAM = \angle KCE$. It implies that $\angle MAH = \angle ECB$. Hence $\angle ALC = \angle AHC = 90°$.